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A note on the two symmetry-preserving covering maps of the gyroid minimal surface

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Abstract. Our study of the gyroid minimal surface has revealed that there are two distinct covering maps from the hyperbolic plane onto the surface that respect its intrinsic symmetries. We show that if a decoration of $\mathbb{H}^2$ is chiral, the projection of this pattern via the two covering maps gives rise to distinct structures in $\mathbb{E}^3$.

PACS. 61.50.Ah Theory of crystal structure, crystal symmetry; calculations and modeling – 89.75.Hc Networks and genealogical trees – 02.20.-a Group Theory – 02.40.-k Geometry, differential geometry, and topology

1 Introduction

This paper is part of a wider project exploring the generation and enumeration of three-dimensional euclidean networks through reticulations of triply periodic minimal surfaces [1–5]. The essential tool is a covering map from the hyperbolic plane, $\mathbb{H}^2$, onto a TPMS, which allows us to transfer patterns in $\mathbb{H}^2$ onto the given surface in $\mathbb{E}^3$. We are then able to systematically generate patterns in $\mathbb{E}^3$ by enumerating patterns in $\mathbb{H}^2$ that are compatible with the covering map. The "patterns" can be tilings, networks, tree-, line-, or sphere- packings — for examples see the papers cited above.

The most commonly studied surfaces in this context are the primitive, P, diamond, D, and gyroid, G, minimal surfaces. They are defined via integrals of a complex Weierstrass function, which for these three surfaces differs only by a complex phase factor (the Bonnet angle). This means the P, D, and G surfaces share the same hyperbolic crystallography [6]. Explicit definitions of covering maps that are fully compatible with the surface symmetries were given in 1989 by Sadoc and Charvolin [7]. We show here that there are in fact two such maps for the G surface.

Such pairs of covering maps from the hyperbolic plane to TPMS are possible only when the intrinsic symmetries of the surface (i.e. the symmetries of the Gauss map and its inverse Weierstrass function) are higher than the extrinsic (euclidean) symmetries. The only known examples where this is the case are the gyroid, discussed here, and lower symmetry relatives, namely the one-parameter families of "rG" (rhombohedral) and "tG" (tetragonal) TPMS [8]. Therefore, we also expect to find two distinct embeddings of a single chiral hyperbolic tiling on the rG and tG surfaces. The possibility of monoclinic and triclinic gyroids remains open. Intriguingly, no further examples of triply periodic surfaces "associate" to TPMS other than the P/D family have been found, despite searches for such associates to the I-WP and C(P) (Neovius) minimal surfaces [9].

Our discovery of two distinct symmetric covers of the gyroid was stimulated by a puzzling example raised by Prof. Michael O'Keeffe. He asked us to explain how two 3-periodic nets (**fcy** and **fcz** in O'Keeffe's naming scheme [10]) that are apparently both reticulations of the G surface can arise from exactly the same hyperbolic tiling. These examples are discussed at the end of the paper. The **fcz** framework has recently been observed as the structure of a crystalline mesoporous germanium oxide, and it is reasonable to expect that the **fcy** relative may be found also [11].

The core result of this paper is that the two maps onto the gyroid give distinct 3-periodic nets only when the hyperbolic tiling is chiral (in a 2d sense). The systematic enumeration of tilings on the gyroid has only recently been undertaken, and this is perhaps why the existence of two covering maps was not noticed earlier.

2 Definition of the covering maps

The P, D, and G surfaces (illustrated in Fig. 1) each have intrinsic surface symmetry related to the $*246$ hyperbolic kaleidoscopic group (we use Conway's notation for 2d orbifolds throughout this paper [12,13]). The $*246$ group is

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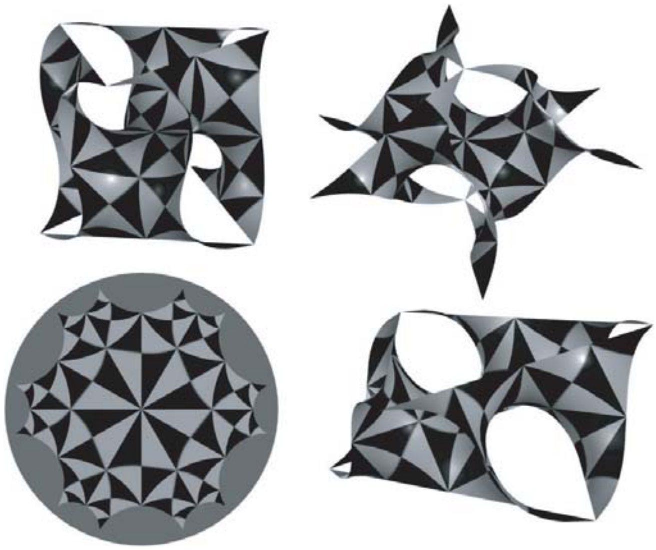

Fig. 1. Clockwise from top left: translational unit cells for the D, G, and P minimal surfaces and their common preimage in the hyperbolic plane. The hyperbolic plane is illustrated using the Poincaré disc model in which straight lines are represented by circular arcs that intersect the disk boundary at right angles. The black and grey triangles are $*246$ fundamental domains — there are 96 such triangles in the dodecagonal translation unit. (This figure is the same as one that appears in [1]).

generated by three reflections, R_1, R_2, and R_3, whose mirror lines bound a triangle in $\mathbb{H}^2$ with corner angles of $\pi/4, \pi/6$, and $\pi/2$. This geometry induces a set of relations for the group: $R_1^2 = R_2^2 = R_3^2 = I$ (the identity) and $(R_1R_2)^2 = (R_2R_3)^4 = (R_1R_3)^6 = I$. Sadoc and Charvolin [7] found that these three closely-related surfaces each have a disk-like fundamental unit in $\mathbb{E}^3$, that pulls back to the same dodecagon in the hyperbolic plane, seen in Figure 1. There are six hyperbolic translations that pair opposite edges of the dodecagon, and generate a normal subgroup, T, of $*246$:

$$\begin{aligned} t_1 &= (R_3R_1R_3R_1R_3R_2)^2 \\ t_2 &= R_3R_1R_3\, t_1\, R_3R_1R_3 \\ t_3 &= (R_1R_3)^2\, t_1\, (R_3R_1)^2 \\ \tau_1 &= (R_3R_1R_2)^2(R_3R_1)^2(R_2R_3R_1)^2 \\ \tau_2 &= R_1R_3R_1\, \tau_1\, R_1R_3R_1 \\ \tau_3 &= R_3\, \tau_1\, R_3. \end{aligned} \qquad (1)$$

These translations satisfy the following identity in $\mathbb{H}^2$:

$$\tau_1 t_2 \tau_3^{-1} t_1^{-1} \tau_2 t_3 \tau_1^{-1} t_2^{-1} \tau_3 t_1 \tau_2^{-1} t_3^{-1} = I.$$

Geometrically, this identity tells us there are twelve dodecagons around each of its vertices. The subgroup, T, generated by the t_i and τ_i translations is isomorphic to the fundamental group of a three-handled torus and has orbifold symbol ooo.

A translational unit cell for the (oriented) G surface is built from the hyperbolic dodecagon by deforming it into a "double pinwheel" shape with six "blades" and two sets of three vertices pinned together. A covering map from the hyperbolic dodecagon onto the G surface can be defined in two distinct ways. At a topological level the existence of two covering maps is seen from the existence of two types of vertex in the fundamental unit: the pinned vertices meeting in two groups of three, and the six vertices at the tips of the blades. This is in contrast to the P and D surfaces where all vertices meet in symmetrically-equivalent pairs. The two covering maps for the G surface correspond to the two possible choices of which vertices lie on the "blades" in the pinwheel and which on the "pins" — see Figure 2.

The essential difference between covering maps of the G surface and the P and D surfaces can also be understood in terms of the Weierstrass integrals. The Bonnet angle for the D is 0°, for the P it is 90° (i.e. the adjoint of the D), and for the G it is[1] $\alpha \approx 38.01°$ (making it an associate of the D and P surfaces). Symmetries of the Weierstrass function mean that setting the Bonnet angle to $\pi - \alpha$ generates a "second version" of the G surface. As subsets of $\mathbb{E}^3$ the two surfaces are identical, but certain straight lines in the complex plane will map to helices of opposite handedness via the two different integrals. In the hyperbolic plane, one such line is that passing through 2-fold points, illustrated in Figure 2. This line maps to a helix following a 3-fold screw axis, with positive handedness in one parametrization of the G surface, and negative in the other.

Although the intrinsic symmetries of the G surface (i.e., the symmetries of the Weierstrass function) pull back to the $*246$ hyperbolic group, the euclidean symmetries of the surface correspond to the rotational 246 subgroup. The space group of the (non-oriented) G surface is $Ia\overline{3}d$, which has a 3-fold and two inequivalent 2-fold rotation axes, $\overline{3}$- and $\overline{4}$- inversion centers, but no mirror planes. The six-fold hyperbolic rotations (conjugate to R_1R_3) map to $\overline{3}$ inversions, four-fold hyperbolic rotations (such as R_2R_3) map to the $\overline{4}$ inversions (whose centers lie on one type of 2-fold axis), while the two-fold hyperbolic rotations (e.g. R_1R_2) map to the other type of 2-fold rotation axes. The $*246$ reflections, R_1, R_2, R_3, do not correspond to euclidean symmetries of the G surface.

The two covering maps $\Phi, \Theta : \mathbb{H}^2 \to G \subset \mathbb{E}^3$ are each defined by a conformal homeomorphism of a patch of the hyperbolic plane onto an asymmetric unit of the surface in $\mathbb{E}^3$ which is then extended by a homomorphism between the 2d hyperbolic and 3d euclidean symmetry groups, $\phi,\ \theta : 246 \to Ia\overline{3}d$ so that for $x \in \mathbb{H}^2$ and $Q \in 246$:

$$\Phi(Qx) = \phi(Q)\Phi(x) \qquad \Theta(Qx) = \theta(Q)\Theta(x). \qquad (2)$$

Swapping the pinned and the blade vertices corresponds to the $*246$ reflection R_1. This gives us a precise relationship between the two covering maps,

$$\Theta(x) = \Phi(R_1(x)). \qquad (3)$$

[1] Precisely: $\alpha = 1/\tan[K'(1/2)/K(1/2)]$, where K is the complete elliptic integral of the first kind.

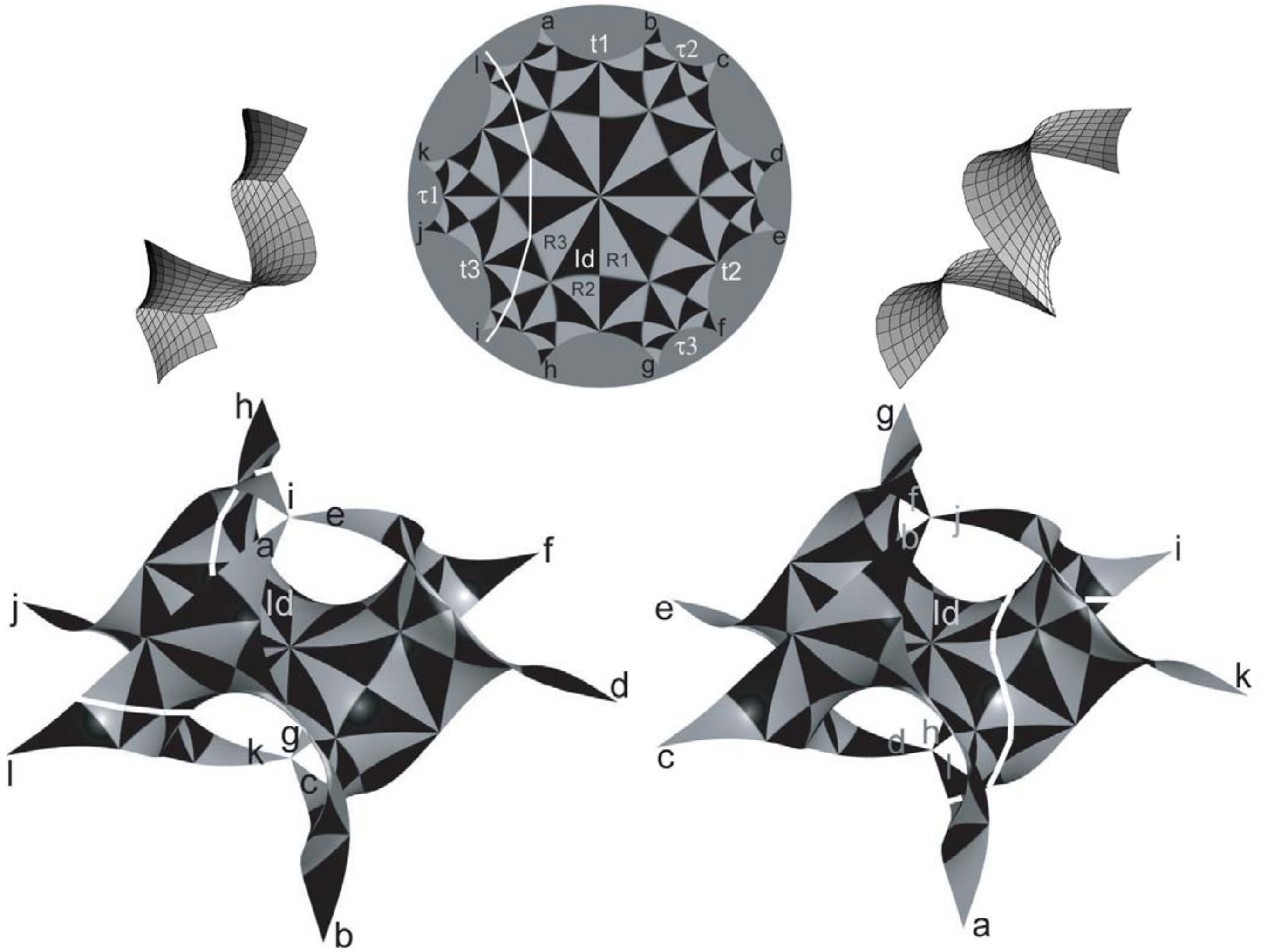

Fig. 2. Top center: the dodecagonal translational patch in the hyperbolic plane has vertex labels a,b,c,...,l and edge labels that denote the image of the central dodecagon by each of the six translations. The white line maps to helices of opposite handedness via the two covering maps. These helices are depicted on the gyroid surfaces below, and as analogously handed helicoidal surfaces at top left and top right. Bottom left: the dodecagon projected onto the G surface via the Φ covering map has the vertex labels given. Bottom right: the dodecagon projected on the G by the Θ covering map switches the pinned and blade vertices.

We next consider the action of the group homomorphisms on the hyperbolic and euclidean translation subgroups. In $\mathbb{H}^2$ we have $T \subset 246 \subset *246$, generated by the six independent hyperbolic translations defined in (1) that pair opposite sides of the dodecagon: $t_1, t_2, t_3, \tau_1, \tau_2, \tau_3$. In $\mathbb{E}^3$ we have three independent translations $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (these are the body-centered rhombohedral lattice vectors) which form an abelian group $L \subset Ia\overline{3}d$. Explicitly the actions of the group homomorphisms ϕ, θ, on the translations are:

$$\begin{array}{ll}
\phi(t_1) = \mathbf{a} & \theta(t_1) = \mathbf{a} \\
\phi(t_2) = \mathbf{b} & \theta(t_2) = \mathbf{c} \\
\phi(t_2) = \mathbf{c} & \theta(t_3) = \mathbf{b} \\
\phi(\tau_1) = -\mathbf{a} - \mathbf{b} & \theta(\tau_1) = \mathbf{a} + \mathbf{b} \\
\phi(\tau_2) = -\mathbf{b} - \mathbf{c} & \theta(\tau_2) = \mathbf{c} + \mathbf{a} \\
\phi(\tau_3) = -\mathbf{c} - \mathbf{a} & \theta(\tau_3) = \mathbf{b} + \mathbf{c}.
\end{array}$$

The group homomorphisms ϕ, θ map T onto L but are not one-to-one — there are subgroups (kernels) $K_\phi, K_\theta \subset T$ which map to the identity in L. These kernels are both isomorphic to the homotopy fundamental group of the G surface, but are distinct normal subgroups of the rotational 246 group. There are six core generators for each kernel (given below) and the full set of infinitely many generators is obtained by conjugating these core elements by each translation in T (i.e. generators are of the form tct^{-1} for $t \in T$ and $c \in K$). For K_ϕ the core generators are:

$$\begin{array}{lll}
\tau_1 t_2 t_1, & \tau_2 t_3 t_2, & \tau_3 t_1 t_3, \\
t_1 t_2 \tau_1, & t_2 t_3 \tau_2, & t_3 t_1 \tau_3;
\end{array}$$

while for K_θ they are:

$$\begin{array}{lll}
\tau_1^{-1} t_3 t_1, & \tau_2^{-1} t_1 t_2, & \tau_3^{-1} t_2 t_3, \\
t_1 t_3 \tau_1^{-1}, & t_2 t_1 \tau_2^{-1}, & t_3 t_2 \tau_3^{-1}.
\end{array}$$

The identifications for the G surface given by Sadoc and Charvolin [7] are elements of the K_ϕ kernel as defined

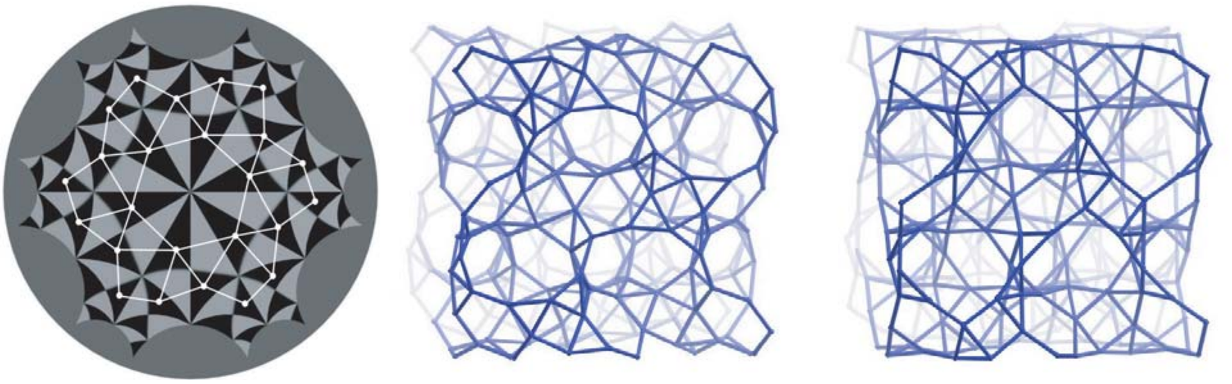

Fig. 3. Left: the dodecagonal translational patch in the hyperbolic plane with a partial sketch of the chiral 5-coordinated net with Schläfli symbol 3.3.4.3.6. Center: the net mapped by the Φ covering map gives **fcy** (viewed slightly off the y-axis). Right: the net mapped via the Θ covering map gives **fcz** (with the same view angle).

above, but T-conjugacies of their six words fail to generate the full kernel.

It is interesting to note that although the two kernels are normal subgroups of 246, they are conjugate within $*246$, that is, $K_\theta = R_i K_\phi R_i$ for $i = 1, 2, 3$. This exposes an error in Section 5 of our previous paper [1]. There we claim that a conjugacy does not affect the covering map. But this is true only if the kernel is left invariant by the conjugacy, which is not the case for the G surface since reflection conjugacies give isomorphic versions of the kernels. The P and D surface covering maps, however, do have kernels that are normal in $*246$, and are therefore unaffected by conjugacy.

3 Achiral $\mathbb{H}^2$ patterns project to isomorphic patterns in $\mathbb{E}^3$

We now show that if a pattern in $\mathbb{H}^2$ is "the same" as its mirror image, then it projects to "the same" pattern on the G surface via the two covering maps. This is intuitively clear from the relationship (3) between the two maps.

Suppose we have a pattern (tiling or decoration) $\mathcal{P}$ in the hyperbolic plane with symmetry group $\Gamma \subset *246$. The pattern is geometrically achiral if its mirror image can be superimposed on the original by an orientation-preserving isometry, i.e., $R(\mathcal{P}) = Q(\mathcal{P})$ for some reflection $R \in *246$ and isometry $Q \in 246$. We can apply the isometry $R_1 R$ to this equation to find that $R_1(\mathcal{P}) = R_1 R Q(\mathcal{P})$, and since the product of two reflections is orientation preserving we have that $R_1 R Q = Q' \in 246$. From the relationship (3) and the definition (2) we now have that

$$\Theta(\mathcal{P}) = \Phi(R_1(\mathcal{P})) = \Phi(Q'(\mathcal{P})) = \phi(Q') \cdot \Phi(\mathcal{P}).$$

Since $\phi(Q')$ is an element of $Ia\overline{3}d$ we have shown that the two projections $\Theta(\mathcal{P})$ and $\Phi(\mathcal{P})$ are isometric.

Conversely, suppose we have that $\Theta(\mathcal{P}) = S \cdot \Phi(\mathcal{P})$ for some isometry of $\mathbb{E}^3$. Since both objects are decorations of the G surface, we have $S \in Ia\overline{3}d$. Therefore, S is the image of some hyperbolic symmetry $Q \in 246$ (actually the K_ϕ coset of Q) under the group homomorphism, i.e., $S = \phi(Q)$. So:

$$\Phi(R_1(\mathcal{P})) = \Theta(\mathcal{P}) = S \cdot \Phi(\mathcal{P}) = \phi(Q) \cdot \Phi(\mathcal{P}) = \Phi(Q\mathcal{P}).$$

Table 1. Crystallographic coordinates for **fcy** and **fcz**. Both have space group $Ia\overline{3}d$, unit cell parameter $a = 5.7366$, and there is one 5-coordinated vertex in Wyckoff position $96h$.

	x	y	z
fcy	0.0945	0.0384	0.8589
fcz	0.0129	0.1284	0.8861

Table 2. The coordination sequences of the 3.3.4.3.6 net in $\mathbb{H}^2$, and its two projections onto the G surface, **fcy** and **fcz**.

net	cs_1	cs_2	cs_3	cs_4	cs_5	cs_6	cs_7
$\mathbb{H}^2$	5	12	24	45	83	155	286
fcy	5	12	24	45	76	106	150
fcz	5	12	24	45	76	109	148

and this implies $R_1(\mathcal{P}) = Q'Q(\mathcal{P})$ for some $Q' \in K_\phi \subset T$, so that $\mathcal{P}$ is indeed geometrically achiral.

This result can be strengthened to show that if a pattern $\mathcal{P}$ is homotopic in $\mathbb{H}^2$ to a geometrically achiral pattern $\mathcal{P}'$, then $\mathcal{P}$ is homotopic to its mirror image, and the corresponding patterns on the G surface, $\Theta(\mathcal{P})$ and $\Phi(\mathcal{P})$ are also homotopic.

We finish the paper with the example nets referred to in the introduction, and illustrated in Figure 3. The hyperbolic net is vertex-transitive, 5-coordinated, has Schäfli symbol 3.3.4.3.6, and symmetry group 246. It is chiral, as can be seen from Figure 3, where the net has vertices only on the black orthoschemes. Thus, the two G surface covering maps project it onto distinct 3-periodic nets called **fcy** and **fcz** in O'Keeffe's labelling scheme [10]. The nets are also sphere packings, with labels 5/3/c41 (**fcy**), and 5/3/c42 (**fcz**) in the enumeration by Fischer [14,15]. In $\mathbb{E}^3$, the two nets both have symmetry group $Ia\overline{3}d$, one kind of vertex and three types of edge, identical geometric density (as homogeneous sphere packings) and the same extended Schläfli vertex symbol (encoding the local ring structure, see [16]) of $3.3.3.4.6.10_4.10_4.10_8.10_8.10_{12}$. The crystallographic coordinates are given in Table 1. The nets are, however, topologically distinct as their coordination sequences show [16], see Table 2. Their sequences differ at shell 6 and the total number of vertices within ten steps of an initial vertex (a measure of topological density) is 1156 (**fcy**) and 1163 (**fcz**). The difference is due to the covering map pinning the white triangles or pinning the black triangles as in Figure 2. Only the black triangles contain vertices of the network, so the different covering maps produce different global topological structure in the networks.

We thank Mike O'Keeffe for bringing this interesting pair of nets to our attention, and Gerd Schröder for helpful discussions about the geometry of the gyroid.

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